Other issues of concern are the signal detection, transmission, and interference. Moreover, path length knowledge and/or control down to a fraction of the wavelength is required for a general interferometric observatory. Whereas these issues are in general tractable for long wavelength imaging applications, they will be especially hard for IR missions. Our observatory is well-suited for long wavelength applications. On the other hand, there are certain technologies that are assumed to exist for the proposed very long baseline, Earth-orbiting observatory to be feasible at visible and IR wavelengths, such as heterodyne or a direct detection method.[7–9] Of particular interest is applying a heterodyne detector to IR applications. Such interest is motivated by NASA's ambitious Origins program.[10] Precurser space-based missions of the Origin's program include the Hubble Space Telescope and Stratospheric Observatory for Far Infrared Astronomy, an IR mission. First- (Space Interferometry Mission), second- (Terrestrial Planet Finder[11]), and third- (Planet Imager[12]) generation missions involve the detection of signals with wavelengths ranging from 0.4–0.9 to 20 μm. Heterodyne detection has several advantages over direct detection, has been demonstrated for an IR application[8] and is the subject of ongoing research.

VI. Conclusions

In this Note, we propose a class of sparse aperture interferometric satellite constellations in Earth orbit that can be used to observe astronomical bodies over the full celestial sphere. This observatory is capable of forming high-resolution images in timescales of a few hours, while completely covering the desired region of the wave number (u–v) plane for a wide range of wavelengths. An optimization procedure is defined that supplies m pixels of resolution with a minimum number of satellites. A lower bound for the minimum number of spacecraft in the constellation is derived, and we show that for the example considered this procedure results in an observatory that is within 0–2 satellites from this lower bound. The zonal J_2 effect is used to scan the observatory across the celestial sphere. Finally, we discuss some practical implementation issues for these observatories.

Acknowledgment

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New, Fast Numerical Method for Solving Two-Point Boundary-Value Problems

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Introduction

IN physics and engineering one often encounters what is called a two-point boundary-value problem (TPBVP). A number of methods exist for solving these problems including shooting, collocation, and finite difference methods.[1,2] Among the shooting methods, the simple-shooting method (SSM) and the multiple-shooting method (MSM) appear to be the most widely known and used methods.

In this Note a new method is proposed that was designed to include the favorable aspects of the SSM and the MSM. This modified simple-shooting method (MSSM) sheds undesirable aspects of these methods to yield a fast and accurate method for solving TPBVPs. The convergence of the modified simple-shooting method is proved under mild conditions on the TPBVP. A comparison of the MSSM, MSM, collocation (CM) and finite difference methods (FDM) is made for a simple example for which all of these methods converge. Further comparison between the MSM and the MSSM can be found in our earlier work,[3] where we studied an optimal control problem with fixed endpoints for a nonlinear system. For that problem it was shown that the MSM failed to converge while the MSSM converged rapidly.

A general TPBVP can be written in the following form:

$$\dot{y}(t) = f(t, y), \qquad a \le t \le b \tag{1}$$

$$r[y(a), y(b)] = 0 \tag{2}$$

where Eq. (2) describes the boundary conditions satisfied by the system. Examples are the familiar initial-value problem (IVP) and first-order necessary conditions obtained by an application of the Pontryagin Maximum Principle in optimal control theory. TPBVPs from optimal control have separated boundary conditions of the type $r_1[y(a)] = 0$ and $r_2[y(b)] = 0$.

Some of the initial publications that deal with TPBVPs are Keller[4,5] and Roberts and Shipman.[1] Provided it converges, the SSM is the simplest, fastest, and most accurate method to solve TPBVPs. However, it is well known that the SSM can fail to converge for problems whose solutions are very sensitive to initial conditions. For such problems, FDM and CM can provide a solution that satisfies the boundary conditions and is close to the actual solution in some sense. This led to the development of the MSM.[6] Morrison et al.[6] first proposed the MSM as a compromise between the SSM and the finite difference methods. Keller[5] refers to the MSM as parallel shooting and also proposed a version of parallel shooting that he called "stabilized march." The FDM and CM schemes are much

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harder to set up than the shooting methods.[1] For nonlinear problems quasi-linearization is used along with finite difference schemes.[7]

In this Note we restrict our attention to problems with no constraints. We compare existing methods with the MSSM in terms of computation times and accuracy of solutions and show that the MSSM is superior. Once a numerical method has converged, the norm of the difference between the desired final states and the actual final states obtained after a reintegration of Eq. (1) is our criterion for accuracy of the solution.

Summary of Existing Methods

As separated boundary conditions are commonly encountered in optimal control, we will consider such boundary conditions in the rest of the Note. However, the methods described next can be extended to more general boundary conditions of the type $F_0 y(a) + F_1 y(b) = \alpha$, where F_0 and F_1 are matrices such that $\text{rank}(F_0) + \text{rank}(F_1) = n$, where n is the length of the vector y. Thus the system under consideration is

$$\dot{y}(t) = f(t, y), \qquad a \le t \le b, \quad y(t) \in \mathbb{R}^n \quad \text{for each} \quad t \tag{3}$$

$$Ay(a) = \alpha, \qquad By(b) = \beta \tag{4}$$

where A and B are $m_A \times n$ and $m_B \times n$ matrices with $m_A = \text{rank}(A)$, $m_B = \text{rank}(B)$, and $m_A + m_B = n$. If $m_A = n$, then we have an IVP. In the presentation that follows, we assume $n \ge 2$ and $m_A < n$. Though not necessary from a theoretical point of view, it is very useful in practice to have $B = [I_{m_B \times m_B} \;\; 0_{m_B \times m_A}]$. This can be achieved by a coordinate transformation that puts B in the preceding form.

Though it is difficult to establish existence and uniqueness for TPBVPs in general, for certain problems that arise from a variational principle (including optimal control problems) one can deduce such properties.[8] As we are interested in numerical methods, we will make some assumptions that ensure the well-posedness of our algorithms.

The remainder of the paper makes use of two assumptions.

Assumption 1: There exists a unique solution to the TPBVP [Eqs. (3) and (4)].

Assumption 2: If $y^*(a)$ denotes the initial condition that leads to the solution of the TPBVP, then there exists a unique solution defined on $[a, b]$ for every initial condition in a sufficiently small neighborhood of $y^*(a)$. Furthermore, the solution is continuously differentiable with respect to changes in the initial condition.

These assumptions imply that the matrix $\partial f(t, y)/\partial y|_{y=y^*(t)}$ exists and is bounded for $t \in [a, b]$, where $y^*(t)$ denotes the optimal solution. The second assumption ensures that numerical methods based on a modified Newton's method will be convergent.

The simple-shooting method transforms a TPBVP into an IVP, where the initial values of selected parameters are varied to satisfy the desired end conditions.[5] A very desirable property of the SSM (provided that it converges) is that the resulting solution is a continuously differentiable function that satisfies Eqs. (3) and (4). This means that the boundary conditions (4) are satisfied when Eq. (3) is integrated over $a \le t \le b$ using the initial condition obtained using the SSM. There can be serious problems with the convergence of the SSM, if the starting initial condition $y(a)$ is not close to $y^*(a)$. As we have no way of knowing $y^*(a)$ beforehand, the SSM is not a practical method for many applications. This drawback of the SSM can be addressed by implementing what is known as the MSM.

The MSM is similar to the SSM, in that one selects unknown parameters at the initial time; however, one does not integrate Eq. (3) all of the way to the final time. Instead, the distance from a corresponding point on a preselected reference path is checked continuously as the integration proceeds, and the integration is aborted when the distance exceeds a tolerance value. Then, one starts the integration again from the corresponding point on the reference path, and the preceding step is repeated, until the system is integrated to the final time. An equation is then formed to match up the discontinuous trajectory segments, and a modified Newton's method is used to reduce the gaps. An advantage of this approach over the SSM is that convergence can now be obtained for a larger class of TPBVPs.[9] A major disadvantage of this method is that the number of parameters to be updated in each iteration can be very large, leading to larger computation times when compared to the SSM (provided that it converges). During each run, one must invert matrices whose row and column dimensions are a linear function of the number of shooting nodes. The number of nodes can be quite large depending on the guesses for the initial unknown parameters. The number of nodes cannot be reduced, even as the guesses improve. Another serious disadvantage of this method is that if the differential equations are reintegrated to result in one continuous trajectory for the system the actual final values might not be close to the desired final values. This is a common problem when solving TPBVPs that result from optimal control because of instability of the systems in the forward direction. For more detail on the multiple shooting method, please refer to Stoer and Bulirsch.[9]

As we mentioned earlier, the FDM and CM are far more complex to set up. For linear systems the FDM transforms the TPBVP to a linear algebra problem. The FDM is often used in conjunction with quasi-linearization for nonlinear systems. A very good discussion of the finite difference schemes can be found in Keller,[5] Roberts and Shipman,[1] whereas a description of the collocation methods can be found in Reinhardt.[2] The collocation method tested in this paper was the bvp4c routine in MATLAB®.[10]

Proposed Modified Simple-Shooting Method

The MSSM combines the attractive quality of convergence for large classes of systems of the MSM, with the satisfaction of the boundary conditions on reintegration property and fast computation times of the SSM. Just like the MSM, there is a choice of a reference path that has the effect of improving convergence. Again, one has to choose unknown parameters at the initial time and integrate the system (3) forward in time, while checking the distance from corresponding points on the reference path. (In some cases it might be more appropriate to integrate backward in time.)[3] The integration is aborted when the distance becomes larger than some tolerance value, just as in the multiple shooting method. Next, the unknown parameters at the initial time are updated so that the trajectory passes through a chosen point on the reference path. This procedure amounts to performing simple shooting on a smaller time interval. This process is repeated as described in the following algorithm on progressively larger intervals of time, until we perform simple shooting on the entire time interval. Thus only the exact number of unknown initial parameters have to be updated in each iteration, which leads to faster convergence. Furthermore, the final trajectory is a solution of the given system, and it exactly connects the initial and final states. This feature of the MSSM contrasts with the MSM, where unavoidable discontinuities can result in significant differences between the desired and actual final state when the resulting approximate solution is used to integrate Eq. (3). The MSSM follows.

Initialization: Choose a distance metric $d(\cdot, \cdot)$ for the space $\mathbb{R}^n$. Next, choose a Lipschitz continuous $\mathbb{R}^n$-valued function $\varphi(t)$ {that is, $d[\varphi(t_1), \varphi(t_2)] \le K_1 |t_1 - t_2|$ for some $K_1 > 0$} that satisfies $A\varphi(a) = \alpha$ and $B\varphi(b) = \beta$. As $\text{rank}(A) = m_A$, the equation $A\varphi(a) = \alpha$ can be solved to obtain m_A of the initial states in terms of the other m_B states that are now treated as parameters. At the kth iteration we denote this parameter vector of length m_B as s_{k-1}.

1) (At step 1:) Choose the parameter vector for the first step $s_0 \in \mathbb{R}^{m_B}$, and compute $\varphi(a)$. The initial vector for step 1 of the algorithm is $y(a) = \varphi(a)$. Denote $t_0 = a$.

2) (At step k:) Solve the system

$$\dot{y}(t) = f(t, y), \qquad a \le t \le b \tag{5}$$

The initial states $y(a)$ are determined from the parameter vector s_{k-1} and the initial condition: $Ay(a) = \alpha$. Denote the solution as $y(t; s_{k-1})$. If $d[y(t; s_{k-1}), \varphi(t)] < \varepsilon$ for $t \in (t_{k-1}, b)$, go to step 4; otherwise, go to step 3.

3) If there exists a $\tilde{t} \in (t_{k-1}, b)$ with $d[y(\tilde{t}; s_{k-1}), \varphi(\tilde{t})] = \varepsilon$, then
   a) Denote $t_k = \tilde{t}$;

Table 1 Comparison of different solution schemes for the simple example

Final time	Solution method	Unknown states $[y_3(0) \quad y_4(0)]$	Time step, s	Error after reintegration	Computation time, s
$t_f = 1$	MSSM	[0.3888 0.3888]	0.1	6.28×10^{-16}	0.02
	MSM	[0.3888 0.3888]	0.1	9.93×10^{-16}	0.15
	CM	[0.3888 0.3888]	10^{-2}	9.39×10^{-3}	0.17
	FDM	[0.3888 0.3888]	10^{-2}	9.47×10^{-3}	0.11
$t_f = 35$	MSSM	[−1.000 −1.000]	0.25	2.4×10^{-3}	1.61
	MSM	[−1.000 −1.000]	0.25	1.2×10^{-2}	3.20
	CM	[−1.000 −1.000]	0.004	24.33	34.41
	FDM	[−0.9992 −0.9992]	0.04	0.161	490.2

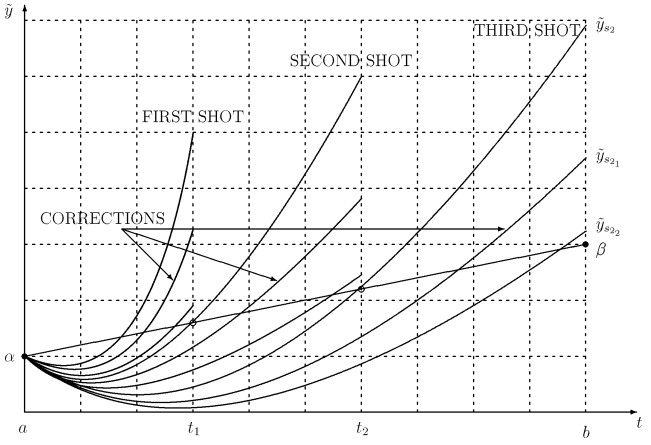

Fig. 1 Illustration of modified simple shooting method.

b) Use a modified Newton's method with cost function $g(s) = d[y(t_k; s), \varphi(t_k)]$ (and tolerance parameter $\varepsilon_1 < \varepsilon$) and find an update to the parameter vector s. Increment k, and go to step 2.

4) If $d[By(b; s_{k-1}), \beta] \geq \delta$ (where $\delta < \varepsilon$), then use a modified Newton's method with cost function $g(s) = d[By(b; s), \beta]$ and tolerance parameter δ. Stop.

There are three parameters ε, ε_1, and δ that must be chosen in addition to the choice of s_0. The parameter ε is chosen such that a SSM converges on the first interval $[t_0, t_1]$. It is possible to choose ε_1 and δ to be equal to each other as long as they are less than ε. In the last step of the MSSM, a SSM is being performed with a starting initial guess s_k that keeps $By(b; s_k)$ close to β. This prevents numerical divergence. Figure 1 illustrates the MSSM. In this case it took three overall shots to integrate from $t = a$ to $t = b$. In this illustration the matrix B is in the form $[I_{m_B \times m_B} \quad 0_{m_B \times m_A}]$.

To analyze the MSSM, we presume that assumptions 1 and 2 stated earlier hold. Let D be a neighborhood of $y^*(a)$ such that the two conditions are satisfied, and let $y_D(t) = \{y(t; a, z) | z \in D\}$ denote the neighborhood around $y^*(t)$ formed by the solutions that start in D. To prove convergence of this method, we have to show that the sequence of stopping times $\{t_k\}$ that is produced by the algorithm converges to b for some finite k. At each of these times, we employ the modified Newton's method as described in step 3b. Therefore we denote the sequence of initial states by $y_{k,l}$, where k corresponds to the stopping time t_k and l corresponds to the subsequence generated by the modified Newton's method. In the following theorem $\mathbf{N}$ represents the set of natural numbers.

Theorem 1: Consider the TPBVP as described in Eqs. (3) and (4), along with the assumptions 1 and 2. Denote the solution to the problem by $y^*(t)$. Suppose that the reference function is chosen such that $\varphi(t) \in y_D(t)$ for each $t \in [a, b]$. Assume that the initial choice s_0 is in D. Then the MSSM results in a sequence $\{t_k\}$; $k \in \mathbf{N}$ such that $t_N = b$ for some finite N. Furthermore, the sequence of initial states $\{y_{k,l}(a)\}$; $k, l \in \mathbf{N}$, converges to $y^*(a)$ in the limit as $\delta \to 0$.

Proof: By our choice of the reference function and initial state, the solutions to the intermediate step $y(t; a, s_{k-1})$ always lie in the set $y_D(t)$. Suppose that the sequence t_k converges to $t^* < b$. Then there exists an $N \in \mathbf{N}$ such that $|t_N - t^*| < \eta$, where η is a positive number that will be specified later. Furthermore, as a result of the modified Newton's method in step 3b, there exists a parameter vector s such that $d[y(t_N; s), \varphi(t_N)] < \varepsilon/6$. By our assumption on the intervals of existence for solutions that start in D, we can extend the solution $y(t; s)$ to $[a, t^*]$. Now the function $y(t; s)$ is a differentiable function of t, and so let K_2 denote its Lipschitz constant on the interval $[a, t^*]$. Also let K_1 denote the Lipschitz constant of the reference function on $[a, b]$. Now suppose that η is chosen so that $\eta = \varepsilon/(6 \max\{K_1, K_2\})$. Then we have

$$\begin{aligned} d[y(t^*, s), \varphi(t^*)] &\leq d[y(t^*, s), y(t_N, s)] + d[\varphi(t^*), \varphi(t_N)] \\ &\quad + d[\varphi(t_N), y(t_N, s)] \\ &< \varepsilon/6 + \varepsilon/6 + \varepsilon/6 \\ &< \varepsilon/2 \end{aligned} \tag{6}$$

Now by the existence of solutions over the interval $[a, b]$ for all initial states in D, we can extend the solution $y(t; s)$ beyond the interval $[a, t^*]$ to an interval $[a, t^* + \mu]$ for some $\mu > 0$. Furthermore, we can choose μ so that $d[y(t^* + \mu; s), \phi(t^* + \mu)] < \varepsilon$ because of inequality (6) and the continuity of $y(t; s)$ and $\phi(t)$ as functions of t. Thus $t^* < b$ cannot be true, and we must have $t^* = b$. The finiteness of N follows because μ can be chosen to be at least η, which depends only on K_1, K_2, and the parameter ε. The last claim follows from the convergence properties of the modified Newton's method.[9] □

Example

In this section we study an example in order to compare and contrast the MSM, MSSM, FDM and CM. The computations were performed in a MATLAB® environment on a standard desktop computer. The comparison was done for the problem (7) with two different final times $t_f = 1$ and 35.

Consider the following system:

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \\ \dot{y}_4(t) \end{bmatrix} = \begin{bmatrix} y_3(t) \\ y_4(t) \\ y_2(t) \\ y_1(t) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1(t_f) \\ y_2(t_f) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{7}$$

where $0 \leq t \leq t_f$. This system was solved with the "bad" initial guess $s_0 = [-100 \quad 2]^T$ with the time step 0.01, $\varepsilon = 2$, and $\varepsilon_1 = \delta = 10^{-3}$. The reference path was chosen to be $\varphi(t) = t[1 \quad 1]^T + [1 \quad 1]^T$. The parameters, initial guess for the unknown initial states, and reference path were kept the same for both the MSM and MSSM.

The computations were carried out for $t_f = 1$ and 35, and the results are reported in Table 1. In the first case ($t_f = 1$) one can see the advantage the MSSM enjoys over the other methods with regard to computation time. The second case ($t_f = 35$) illustrates the accuracy of the MSSM clearly. Only the first few significant digits for the states are displayed in Table 1, though the computations were performed in double precision. The integration method employed

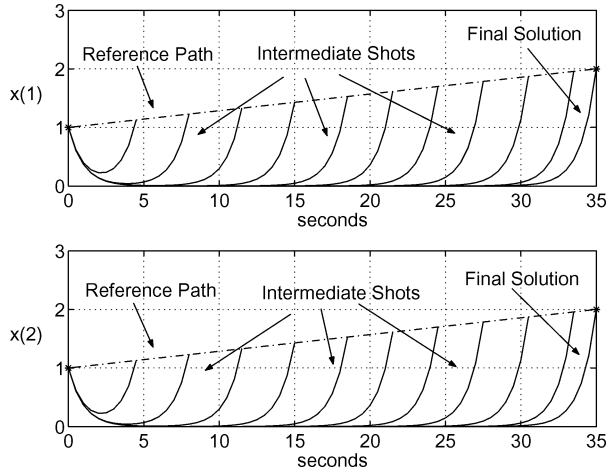

Fig. 2 MSSM applied to the linear system example.

for the MSSM, MSM, and CM was the classical fourth-order Runge–Kutta method. Figure 2 shows the result of the MSSM including the reference path and the intermediate trajectories. The SSM failed for this problem (with $t_f = 35$), while both the MSM and MSSM were successful.

The collocation method tested was the bvp4c routine in MATLAB®. For this method the mesh selection is based on the residual of the C^1 continuous solution that is fourth-order accurate uniformly in the time interval.[10] For $t_f = 1$ we chose 101 uniformly spaced nodes corresponding to a time step of 0.01 s as the initial mesh. For $t_f = 35$ a uniform mesh corresponding to a time step of 0.004 s was chosen at the initial step.

Conclusions

A new method for solving two-point boundary-value problems has been presented. An example was provided that clearly illustrates that the MSSM results in an accurate solution that takes significantly less computation time than the MSM, finite difference, and collocation methods.

Among its desirable features are that it requires the inversion of much smaller matrices than those required to be inverted in the MSM, finite difference, and collocation methods. Another fact that makes the MSSM more appealing is that the solution results in a trajectory that satisfies the system differential equations. This property is very important in optimal control problems where the systems are unstable in forward time. The MSM, finite difference, and collocation methods do not share this property with the MSSM. Because of the instability of many systems in the forward direction, these other methods can lead to erroneous solutions, in the sense that the solution trajectory on reintegration does not satisfy the boundary conditions at the final time.

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Approximate Analytical Criterion for Aircraft Wing Rock Onset

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I. Introduction

THERE is continuing interest in analytical criteria for aircraft departure because of the physical insight they provide into the associated instability phenomena and their use in identifying critical aerodynamic parameters related to departure susceptibility. However, exact analytical criteria are usually too complicated to be of any use; hence, considerable effort has been made over the years to arrive at approximate analytical criteria for departure phenomena that may be usefully employed early in the aircraft design process. Initial work by Bryant and Zimmerman (see Ref. 1) established approximate criteria for static directional instability, also called directional divergence or yaw departure. Dynamic directional instability, on the other hand, is associated with unstable Dutch roll oscillations linked to onset of a phenomenon called wing rock.[2] An exact analytical criterion for wing rock onset, based on the vanishing of the Routh discriminant, is attributed to Duncan[3]; however, the resulting expression is too unwieldy to provide any physical insight and, not surprisingly, has found little use. [The Routh discriminant is given by $R = D(BC - D) - EB^2 = 0$, where B, C, D, and E are the coefficients of the lateral–directional characteristic polynomial, $P_{\text{lat}}(\lambda) = \lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E$.] Instead, Moul and Paulson[4] came up with an approximate criterion called "dynamic directional instability parameter" or $C_{n_{\beta\text{dyn}}}$, but attempts[5,6] to relate rigorously the $C_{n_{\beta\text{dyn}}}$ parameter to an instability phenomenon led to the conclusion that $C_{n_{\beta\text{dyn}}}$ was, in fact, only an approximate static directional instability criterion. Efforts to derive a useful, approximate version of the Routh discriminant criterion for wing rock onset have been ongoing (e.g., as in Ref. 7), whereas, $C_{n_{\beta\text{dyn}}}$, by virtue of being an important component of the Routh discriminant, has de facto found

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